

# Quantum-Classical Correspondence of Dynamical Observables, Quantization and the Time of Arrival Correspondence Problem<sup>\*</sup>

Eric A. Galapon<sup>†</sup>

Theoretical Physics Group, National Institute of Physics  
University of the Philippines, Diliman Quezon City, 1101 Philippines

February 1, 2008

## Abstract

We raise the problem of constructing quantum observables that have classical counterparts without quantization. Specifically we seek to define and motivate a solution to the quantum-classical correspondence problem independent from quantization and discuss the general insufficiency of prescriptive quantization, particularly the Weyl quantization. We demonstrate our points by constructing time of arrival operators without quantization and from these recover their classical counterparts.

## 1 Introduction

It is generally believed that classical mechanics is the contraction of quantum mechanics in some appropriate limit of vanishing  $\hbar$  [1]. Thus in principle every classical observable—e.g. a real-valued function in the phase space—is the contraction of some quantum observable. However, quantum observables are generally constructed by the quantization of classical observables, the mapping of the algebra of functions

---

<sup>\*</sup>Invited talk at the ICQO 2000, Raubichi, Belarus, May 28-31, 2000, *Opt. & Specs* **91** (2001) 429.

<sup>†</sup>email: egalapon@nip.upd.edu.ph

in the phase space to the algebra of self-adjoint operators acting in some Hilbert space [2]. Obviously this introduces circularity when one invokes the correspondence principle. This is unsatisfactory if quantum mechanics were to be internally coherent and autonomous from classical mechanics.

This glaring unsatisfactory state of quantum affair has been ignored because quantization has been successful in most of its practical applications in the first place. One can, for example, mention its convincing accuracy in predicting atomic spectra by the quantization of the classical Hamiltonian of the classically interacting charged particles. This success has encouraged the development of numerous theories in quantization, such as geometric and deformation quantizations among others. It is well-known though that there are obstructions to quantization [3]. For example, in flat Euclidean space, Groenwold and van Hove [4] have shown that there exists no quantization of all classical observables consistent with Schrodinger's quantization scheme. On the other hand, prescriptive mapping of classical observables to quantum observables, such as Weyl prescription, is known to be generally inconsistent with Dirac's "Poisson bracket-commutator correspondence".

The existence of obstructions to quantization inevitably limits the class of dynamical observables that can be consistently quantized. This sobering handicap and the circularity of quantization force upon us to re-examine the problem of constructing quantum observables corresponding to the classical ones. In this paper we raise the problem of constructing quantum observables that have classical counterparts without quantization. Specifically we seek to accomplish the following: Define and motivate a solution to the quantum-classical correspondence problem independent from quantization, and discuss the general insufficiency of prescriptive quantization, particularly the Weyl quantization, to satisfy some axioms of quantum mechanics. We achieve our purpose by addressing the the time of arrival quantum-classical correspondence problem—the problem of constructing time of arrival operators which is consistent with the correspondence principle. The paper is organized as follows. In Section-2 we define what we mean by the quantum-classical correspondence of dynamical observables and introduce the concept of supraquantization; in Section-3 we outline the solution to the time of arrival QCC-problem; in Section-4 we outline the solution to the linear potential, the harmonic oscillator and the quartic oscillator; in Section-5 we compare Weyls' quantization and the result of our treatment; finally, in Section-6 we outline questions raised by the present report. Our treatment is at the formal level. Its rigorous version shall be given elsewhere.

## 2 The Quantum-Classical Correspondence Problem

The quantum-classical correspondence (QCC) problem is generally understood as the problem of deriving the quantum image of a given classical system and from the quantum image recover classical mechanics via an appropriate limiting procedure involving the vanishing of the Planck's constant [6]. The QCC problem has three aspects, the correspondence between classical and quantum states, dynamics, and observables. In this paper we limit ourselves to the quantum-classical correspondence problem of dynamical observables. Thus we address the issue of deriving the quantum image of a classical observable and recovering the same classical observable from its quantum image.

Any solution to the QCC-problem for a given class of observables consists of (i) a prescription of obtaining the corresponding quantum observables, and (ii) a mapping of these to their respective classical counterparts, i.e. an implementation of the correspondence principle [6]. A satisfactory solution should solve the QCC-problem without conflict with the rest of quantum mechanics. At present quantum observables are constructed by the method of quantization, a prescriptive mapping of the classical observables to quantum operators in a Hilbert space [2]; the classical limits of these are obtained by the inversion of the prescribed mapping. However, this particular solution has several unsatisfactory features as mentioned above, most notorious is their general incompatibility with the required values of commutators as it shall be demonstrated below for the Weyl quantization for the Hamiltonian-time of arrival pair [4].

If we seek a more satisfactory solution, then we have to drop quantization and start elsewhere. But where? The earlier work of Mackey [5] provides us with the motivation to seek solution within quantum mechanics itself. We recall that the quantization of free particle in one dimension is accomplished by promoting its position and momentum  $(q, p)$  into the operators  $(\hat{q}, \hat{p})$  and their Poisson bracket  $\{q, p\} = 1$  into the commutator relation  $[\hat{q}, \hat{p}] = i\hbar$ , and the energy  $H = (2\mu)^{-1}p^2$  into the Hamiltonian operator  $\hat{H} = (2\mu)^{-1}\hat{p}^2$ . Mackey's work obviates these quantization prescriptions by starting not from the classical description but from the axioms of quantum mechanics and the property of free space. Starting from the basic axiom that the proposition for the location of the particle in different volume elements are compatible and the fundamental homogeneity of free space, one derives the position and the momentum operators together with the canonical commutation relation they satisfy. On the other hand, requiring Galilean invariance in the lattice of propositions, one derives the free quantum Hamiltonian.

The free particle provides an excellent example of the existence of more than one

solution to the quantum-classical-correspondence problem at the level of deriving the quantum image of a given set of classical observables. While both solutions yield similar results, they differ in many respects, thus distinct from one another. The former presupposes classical mechanics, while the latter upholds the autonomy of quantum mechanics; the former introduces circularity when invoking the correspondence principle, while the latter sanctions the correspondence principle as a legitimate consequence of the acknowledged preponderance of quantum mechanics over classical mechanics. Should we make preference on one solution over the other? If we required internal coherence and autonomy of quantum mechanics, then the answer is yes. We are then obliged to accept the latter in favor of the former. And seek solution in general to the quantum-classical correspondence problem of its kind.

The former solution is an example of quantization. The latter is an example of what we shall refer to as *supraquantization* ( for the sake of differentiation). Supraquantization because it is beyond quantization. Quantization is the derivation of the quantum observable corresponding to a given classical observable by means of a specified mapping of the c-valued observable to an operator valued observable. Supraquantization is the derivation of quantum observable corresponding to a given classical observable without quantization. Quantization presupposes the axioms of classical mechanics. Supraquantization, on the other hand, presupposes the axioms of quantum mechanics including the postulated properties of the physical universe and other principles. Then, by definition, Mackey's construction of the position and momentum operators together with their commutation relation and the Hamiltonian is a supraquantization of their classical counterparts. Mackey's construction required the axiom of compatibility of the propositions for the location of the particle, the property of homogeneity of free space, and the principle of Galilean invariance.

In both methods of obtaining quantum observables, the classical observable plays two different roles. In quantization, it is the starting point; while in supraquantization, it is a boundary condition. The correspondence principle requires that if a quantum observable corresponds to a classical observable, then the former should reduce to the latter in the limit of vanishing  $\hbar$ . Then if supraquantization gives the correct quantum observable, then that observable should approach its classical limit. As a consequence of the role of the classical observable as a boundary condition, supraquantization breaks the vicious circle inherent in the quantization procedure.

In the following we demonstrate everything we have said about supraquantization by addressing the time of arrival quantum-classical-correspondence (TOA-QCC) problem. We shall limit ourselves in one dimension. The solution to the TOA-QCC problem should contain the prescription of deriving the time of arrival operator and the prescription of obtaining the classical time of arrival. The second part has a

standard solution and it can now be given. Let  $\langle q | \hat{A} | q' \rangle$  be the configuration space kernel of the operator  $\hat{A}$ , then the classical limit of  $\hat{A}$  is given by

$$A(q, p) = \lim_{\hbar \rightarrow 0} 2\pi \int_{-\infty}^{\infty} \left\langle q + \frac{v}{2} \left| \hat{A} \right| q - \frac{v}{2} \right\rangle \exp\left(-i \frac{vp}{\hbar}\right) dv \quad (1)$$

whenever the limit exists [11]. So when one knows how to obtain the kernel of the time operator, equation (1) provides the transition to the classical regime.

### 3 The Time of Arrrival Quantum-Classical Correspondence Problem

#### 3.1 The Classical Time of Arrival

Consider a particle with mass  $\mu$  in one dimension whose Hamiltonian is  $H(q, p)$ . If the state of the particle at time  $t = 0$  is given by the point  $(q, p)$  in the phase space, what is the time,  $T_x$ , that it will arrive at the point  $q(t = T_x) = x$ ? The solution to this problem is straightforward and is given by

$$T_x(q, p) = \text{sgn}(p) \sqrt{\frac{\mu}{2}} \int_q^x \frac{dq'}{\sqrt{H(q, p) - V(q')}} \quad (2)$$

whenever the integral exists and is real valued. (For a derivation of equation (2), see ref-[8].) For a given energy  $H(q, p)$ , the region in the phase space in which equation (2) exists and real valued is the classically accessible region to the particle. Because  $T_x(q, p)$  is a time interval, it is canonically conjugate to the Hamiltonian, i.e.  $\{H, T_x\} = 1$ . By virtue of  $T_x$ 's dependence on the phase space points  $(q, p)$ ,  $T_x$  is a dynamical observable.

It is our objective to derive equation (2) from a quantum observable. But before we can address this problem, let us define the local time of arrival,  $t_x(q, p)$ , at a given point  $x$  as the time of arrival at  $x$  in some small neighborhood of  $q$ . The local time of arrival is technically the expansion of equation (2) about the free time of arrival at  $x$ . For a given Hamiltonian  $H = (2\mu)^{-1}p^2 + V(q)$ , the local time of arrival is given by

$$t_x(q, p) = \sum_{k=0}^{\infty} (-1)^k T_k(q, p; x) \quad (3)$$

Potential $V(q)$	Local Time of Arrival $-t_0(q, p)$	Global Time of Arrival $-T_0(q, p)$
$\lambda q$	$\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^k (k+1)! k!} \mu^{k+1} \lambda^k \frac{q^{k+1}}{p^{2k+1}}$	$\frac{p}{\lambda} \left( \left( 1 + 2 \frac{\mu \lambda q}{p^2} \right)^{\frac{1}{2}} - 1 \right)$
$\frac{1}{2} \omega^2 \mu q^2$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k + 1} \mu^{2k+1} \omega^{2k} \frac{q^{2k+1}}{p^{2k+1}}$	$\frac{1}{\omega} \tan^{-1} \left( \frac{\mu \omega q}{p} \right)$
$\lambda q^4$	$\frac{\Gamma(\frac{3}{4})}{8\pi^{-\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{(-2)^{k+1} \Gamma(-k - \frac{1}{4})}{\Gamma(\frac{1}{2} - k)} \mu^{k+1} \lambda^k \frac{q^{4k+1}}{p^{2k+1}}$	$\pm \sqrt{\frac{\mu}{2}} \int_0^q \frac{dq'}{\sqrt{H(q, p) - \lambda q'^4}}$

Table 1: The local and global times of arrival at the origin for the linear potential, harmonic oscillator and quartic oscillator.

where the  $T_k(q, p; x)$ 's are determined by the following recurrence relation:  $T_0(q, p; x) = \mu p^{-1}(x - q)$  and

$$T_k(q, p; x) = \frac{\mu}{p} \int_q^x \frac{dV}{dq'} \frac{\partial T_{k-1}(q', p; x)}{\partial p} dq' \quad (4)$$

for  $k > 0$ . It can be shown that if  $p \neq 0$  and if  $V$  is continuous at  $q$ , then there exists a neighborhood of  $q$  determined by the neighborhood  $|V(q) - V(q')| < K_\epsilon \leq (2\mu)^{-1} p^2$  such that for every  $x$  in the said neighborhood of  $q$ ,  $t_x(q, p)$  converges absolutely and uniformly to  $T_x(q, p)$ .

Because  $T_x(q, p)$  is defined in the entire accessible region of the particle, we shall refer to it as the global time of arrival.  $t_x(q, p)$  converges to  $T_x(q, p)$  only in a small neighborhood so that  $T_x(q, p)$  is the analytic extension of  $t_x(q, p)$ . In the following sections we show that the local TOA, and thus the global TOA by extension, can be determined completely from pure quantum mechanical consideration. We shall limit though our discussion for the time of arrival at the origin. Table-1 lists the local and the global times of arrival at the origin for the linear potential, the harmonic oscillator, and the quartic oscillator. The local times of arrival are arrived recursively using equation (4).

### 3.2 Supraquantization of the Classical Time of Arrival

Now we proceed in solving for the time of arrival quantum-classical correspondence by supraquantization. We shall proceed formally. Consider the Hilbert space  $\mathcal{H} = L^2(\mathfrak{R}, dq)$  and its particular rigging  $\Phi^\times \supset \mathcal{H} \supset \Phi$  where  $\Phi$  is the fundamental space

of all infinitely differentiable complex valued functions with compact support in the configuration space  $\mathfrak{R}$ . Given the self-adjoint Hamiltonian  $\hat{H}$  which leaves  $\Phi$  invariant under its action, we define its rigged Hilbert space extension,  $\hat{H}^\times$ , in the entire  $\Phi^\times$  as the operator  $\langle \hat{H}^\times \phi | \varphi \rangle = \langle \phi | \hat{H} \varphi \rangle$  for all  $\phi \in \Phi^\times$  and  $\varphi \in \Phi$ . The condition on the invariance of  $\Phi$  under the Hamiltonian restricts us to infinitely differentiable potentials which includes entire analytic potentials. This is not a severe restriction because we can always choose a different rigging of the Hilbert space to accommodate other potentials. In the following whenever refer to the Hamiltonian we mean its rigged Hilbert space extension and shall continue to denote it by  $\hat{H}$ .

For a given Hamiltonian  $\hat{H}$  whose rigged Hilbert space extension is explicitly given by

$$\hat{H}\phi = -\frac{\hbar^2}{2\mu} \frac{d^2\phi}{dq^2} + V(q)\phi \quad \text{for all } \phi \in \Phi^\times, \quad (5)$$

we assert that the corresponding time of arrival operator is given by the operator  $\hat{T} : \Phi \mapsto \Phi^\times$  whose explicit action on  $\Phi$  is

$$(\hat{T}\varphi)(q) = \int_{-\infty}^{\infty} \langle q | \hat{T} | q' \rangle \varphi(q') dq'. \quad (6)$$

in which the kernel must be symmetric, i.e.  $\langle q | \hat{T} | q' \rangle = \langle q' | \hat{T} | q \rangle^*$ . The solution then depends on our ability to construct the time kernel,  $\langle q | \hat{T} | q' \rangle$ , for a given Hamiltonian. Once the time kernel  $\langle q | \hat{T} | q' \rangle$  has been solved, the classical time of arrival operator can then be recovered by means of equation (1), specifically

$$T_0(q, p) = \lim_{\hbar \rightarrow 0} 2\pi \int_{-\infty}^{\infty} \left\langle q + \frac{v}{2} \left| \hat{T} \right| q - \frac{v}{2} \right\rangle \exp\left(-i \frac{vp}{\hbar}\right) dv \quad (7)$$

We will find below that under some conditions equation (7) reproduces exactly the local time of arrival and, thus, the global one by extension.

But how do we determine the kernel  $\langle q | \hat{T} | q' \rangle$  without resorting to quantization? We accomplish this in two steps. First is by specifying the quantum evolution of  $\hat{T}$ . In Heisenberg's representation, the time evolution of a quantum observable  $\hat{A}(t) = e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t}$  is governed by  $i\hbar \dot{\hat{A}}(t) = (\hat{A}(t)\hat{H} - \hat{H}\hat{A}(t))$ , in some appropriate domain. If  $\hat{A}(t)\Phi \mapsto \Phi^\times$  for all  $t$  and  $\Phi$  is invariant under  $\hat{H}$ , then the evolution equation can be formally defined in the entire  $\Phi$ . Now if  $\hat{T}$  is the time of arrival operator, then it must evolve according to  $\dot{\hat{T}}(t) = -\hat{I}$ . Imposing this condition we arrive at the canonical commutation relation

$$([\hat{H}, \hat{T}]\varphi)(q) = i\hbar \varphi(q) \quad \text{for all } \phi \in \Phi \quad (8)$$

satisfied by the Hamiltonian and the time of arrival operator. Equation (8) is the basic condition satisfied by  $\hat{T}$  but it is not sufficient to completely determine  $\hat{T}$ .

The second step is by employing a transfer principle. We hypothesize that each element of a class of observables, such as the time of arrival, shares a common set of properties with the rest of its class such that when a particular property is identified for a specific element of the class that property can be transferred to the rest of the class without discrimination.

We exploit this in determining the kernel  $\langle q | \hat{T} | q' \rangle$  by solving the simplest in the class of time of arrival observables, the free particle. We start by recalling that the free particle is Galilean invariant, a consequence of the homogeneity of free space. It will not matter then where we place the origin. This implies that the commutation relation (8) holds independent of the choice of origin. Because of this and because the free Hamiltonian is Galilean invariant, we require that the time kernel for the free particle must itself be Galilean invariant. Specifically if  $t_a$  is translation by  $a$ , i.e.  $t_a(q) = q + a$  and if  $\langle q | \hat{T} | q' \rangle$  is the free particle kernel, then the translated free time of arrival operator  $\hat{T}_a = \int dq \langle t_a(q) | \hat{T} | t_a(q') \rangle$  must still satisfy equation (8). In addition to translational invariance,  $\langle q | \hat{T} | q' \rangle$  must be symmetric, and it must be chosen such that equation (8) is satisfied given the free Hamiltonian  $\hat{H}\phi = -\hbar^2(2\mu)^{-1}\phi''$ , and it must reproduce the free time of arrival at the origin via equation (7). A solution satisfying all these requirements is given by

$$\langle q | \hat{T} | q' \rangle = \frac{\mu}{i 4 \hbar} (q + q') \text{sgn}(q - q'). \quad (9)$$

We note though that (9) is not unique. The kernel  $\hbar^{-1}\mu |a - a'|$  is dimensionally consistent with (9) and it is Galilean invariant and it commutes with the free Hamiltonian in the entire  $\Phi$ . Moreover it vanishes in the classical limit. Then real factors of it can be added to (9) without sacrificing any of the required properties of the free particle kernel. However,  $\hbar^{-1}\mu |a - a'|$  arises only because of Galilean invariance which is an exclusive property of the free particle. Since we are aiming at exploiting the assumed transfer principle, we can not carry it over to the rest of its class. We shall have more to say about this latter.

Having solved the free particle kernel, we proceed in implementing the transfer principle. We hypothesize that all time kernels assume the same form. Thus, from equation (9), we assume that the time kernel is given by

$$\langle q | \hat{T} | q' \rangle = \frac{\mu}{i \hbar} T(q, q') \text{sgn}(q - q') \quad (10)$$

where  $T(q, q')$  depends on the given Hamiltonian. Inferring from the free particle time kernel, we require that  $T(q, q')$  be real valued, symmetric,  $T(q, q') = T(q', q)$ ,



and analytic. We determine  $T(q, q')$  by imposing condition (8) on  $\hat{T}$ . Substituting equation (10) back into the left hand side of equation (8) and performing two successive integration by parts, we find that  $T(q, q')$  must satisfy the partial differential equation, which we shall refer to as the time kernel equation,

$$-\frac{\hbar^2}{2\mu} \frac{\partial^2 T(q, q')}{\partial q^2} + \frac{\hbar^2}{2\mu} \frac{\partial^2 T(q, q')}{\partial q'^2} + (V(q) - V(q')) T(q, q') = 0 \quad (11)$$

and the boundary condition

$$\frac{dT(q, q)}{dq} + \frac{\partial T(q', q')}{\partial q} + \frac{\partial T(q, q)}{\partial q'} = 1. \quad (12)$$

for all  $q, q' \in \mathbb{R}$ . The boundary condition (12) defines a family of operators canonically conjugate to the extended Hamiltonian in the sense required by equation (8). This is a reflection of the fact that there are numerous operators that are canonically conjugate to a given Hamiltonian. We fix the boundary condition by referring to the free particle. The  $T(q, q')$  of the free particle is  $\frac{1}{4}(q + q')$ . By inspection  $T(q, q')$  satisfies both (11) and (12). Again invoking the assumed transfer principle, we arrive at the following boundary conditions for the time of arrival at the origin,

$$T(q, q) = \frac{q}{2}, \quad T(q, -q) = 0. \quad (13)$$

We claim that equations (11) and (13) constitute the supraquantization of the local time of arrival consistent with the correspondence principle. We will demonstrate this claim below. (Other methods of constructing time of arrival operator without quantization are given in references [7] and [8]. Their motivation, however, is different from ours.)

We remark that our ability to extract probability distributions from the time of arrival operators defined by equation (6) depends on whether the operators (6) have nontrivial Hilbert space projections. If their restrictions on the Hilbert space exists, then they will be generally unbounded and non-self-adjoint. Then they will be classified as POV-observables. Finding the spectral resolution of the time of arrivals is a non-trivial problem and it shall not be addressed here. This will be the topic of a more extensive and indepth treatment of the time of arrival quantum classical correspondence problem.

## 4 Explicit Examples

Table-2 summarizes the solution to the time kernel equation for the linear potential, the harmonic oscillator and the quartic oscillator. We have arrived at the solutions

Potential $V(q)$	Solution to the Time Kernel Equation $T(q, q')$
$\lambda q$	$\frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left( \frac{\mu\lambda}{4\hbar^2} \right)^k (q+q')^{k+1} (q-q')^{2k}$
$\frac{1}{2}\omega^2 \mu q^2$	$\frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \frac{\mu\omega}{2\hbar} \right)^{2k} (q+q')^{2k+1} (q-q')^{2k}$
$\lambda q^4$	$\frac{1}{4} \sum_{m=0}^{\infty} \sum_{n=2m}^{\infty} \Delta_{m,n} \left( \frac{\mu\lambda}{8\hbar^2} \right)^{n-m} (q+q')^{4n+1-6m} (q-q')^{2n}$

Table 2: The solution to the time kernel equation for the linear potential, harmonic oscillator and quartic oscillator.

by changing variables from  $(q, q')$  to  $(u = q + q', v = q - q')$ . The differential equation (11) and the boundary condition then assume the form

$$-2\frac{\hbar^2}{\mu} \frac{\partial^2 T}{\partial u \partial v} + \left( V\left(\frac{u+v}{2}\right) - V\left(\frac{u-v}{2}\right) \right) T(u, v) = 0 \quad (14)$$

$$T(u, 0) = \frac{u}{4}, \quad T(0, v) = 0 \quad (15)$$

We have sought an analytic solution in powers of  $u$  and  $v$ , i.e.  $T(u, v) = \sum_{m,n \geq 0} \alpha_{m,n} u^m v^n$  where the  $\alpha_{m,n}$ 's are constants determined by equations (14) and (15) for a given potential. Once the solution to equation (14) for a given potential has been found, we transform back to  $(q, q')$  to get the results in Table-2.

Before we proceed let us define the following transform of the time kernel,

$$\mathcal{T}_{\hbar}(q, p) = 2\pi \int_{-\infty}^{\infty} \left\langle q + \frac{v}{2} \left| \hat{T} \right| q - \frac{v}{2} \right\rangle \exp\left(-i\frac{vp}{\hbar}\right) dv. \quad (16)$$

$\mathcal{T}_{\hbar}(q, p)$  is real valued and odd with respect to  $p$ . First let us consider the linear potential and the harmonic oscillator. Both systems have linear classical equations of motion, or simply linear systems. Calculating  $\mathcal{T}_{\hbar}(q, p)$  for these systems on using the identity

$$\int_{-\infty}^{\infty} \sigma^{m-1} \text{sgn}(\sigma) \exp(-ix\sigma) d\sigma = \frac{(m-1)!}{i^m \pi} x^{-m}, \quad (17)$$

it is straightforward to show that it exactly reproduces the local time of arrival at the origin as given by Table-1, i.e.

$$\mathcal{T}_h(q, p) = t_0(q, p) \quad (18)$$

for both systems.

We turn to the quartic oscillator. In comparison with the previous two examples, it is nonlinear in the sense that it has nonlinear classical equation of motion. The  $\Delta_{m,n}$ 's in the solution are constants satisfying the recurrence relation  $(4n + 1 - 6m)n\Delta_{m,n} = \Delta_{m,n-1} + \Delta_{m-1,n-2}$  with  $\Delta_{0,0} = 1$ , for  $n \geq 2m$ ,  $m, n \geq 0$ . For our present purposes we don't gain anything by writing down explicitly the  $\Delta_{m,n}$ 's. It is sufficient to give the value of  $\Delta_{0,n} = (4^{n+2}n! \Gamma(\frac{3}{4}))^{-1} (-1)^n \Gamma(-\frac{1}{4} - n)$ . Calculating for its  $\mathcal{T}_h(q, p)$ , we find

$$\begin{aligned} \mathcal{T}_h(q, p) &= 2\mu \sum_{m=0}^{\infty} \hbar^{2m} \sum_{n=2m}^{\infty} (-1)^n \Delta_{m,n} (2n)! \left(\frac{\mu\lambda}{8}\right)^{n-m} \frac{(2q)^{4n+1-6m}}{p^{2n+1}} \\ &= \frac{\Gamma(3/4)}{8\pi^{-\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-2)^{k+1} \Gamma(-\frac{1}{4} - k)}{\Gamma(\frac{1}{2} - k)} \mu^{n+1} \lambda^n \frac{q^{4n+1}}{p^{2n+1}} + \mathcal{O}(\hbar^2) \\ &= t_0(q, p) + \mathcal{O}(\hbar^2) \end{aligned} \quad (19)$$

We see that equation (19) reduces only to the classical time of arrival in the limit of vanishing  $\hbar$  or infinitesimal  $\hbar$ .

The above examples are special cases of the following general result for entire analytic potentials,

$$\mathcal{T}_h(q, p) = \begin{cases} t_0(q, p) : \text{linear systems} \\ t_0(q, p) + \mathcal{O}(\hbar^2) : \text{non-linear systems} \end{cases} \quad (20)$$

For these systems  $\lim_{\hbar \rightarrow 0} \mathcal{T}_h(q, p) = t_0(q, p)$ , and to  $T_0(q, p)$  by extension, in keeping with the correspondence principle. We can then claim that, at the formal level, equations (6) and (7) constitute a solution to the time of arrival quantum-classical correspondence problem without quantization.

## 5 The Insufficiency of Weyl Quantization

Weyl quantization is a prescription which maps a given classical dynamical observable into a quantum observable [10]. It is based on some symmetric ordering of non-commuting observables. Let  $f(q, p)$  be an observable defined on the phase

space. Weyl quantization maps  $f(q, p)$  into the operator  $\hat{\mathbf{W}}_{[f]}$ .  $\hat{\mathbf{W}}_{[f]}$  acts on the configuration space functions  $\psi$  according to

$$(\hat{\mathbf{W}}_{[f]}\psi)(q) = \int_{-\infty}^{\infty} \langle q | \hat{\mathbf{W}}_{[f]} | q' \rangle \psi(q') dq' \quad (21)$$

where the kernel is

$$\langle q | \hat{\mathbf{W}}_{[f]} | q' \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} f\left(\frac{q+q'}{2}, p\right) \exp\left[\frac{i}{\hbar}(q-q')p\right] dp \quad (22)$$

Weyl quantization has the remarkable property that it is translationally invariant. Given the kernel of  $\hat{\mathbf{W}}_{[f]}$  one recovers the classical observable by a inversion of the Weyl transform,

$$f(q, p) = 2\pi \int_{-\infty}^{\infty} \left\langle q + \frac{v}{2} \left| \hat{\mathbf{W}}_{[f]} \right| q - \frac{v}{2} \right\rangle \exp\left(-\frac{i}{\hbar}vp\right) dv \quad (23)$$

Equations (22) and (23) establish a one-to-one correspondence between classical observables and their quantum counterparts via Weyl's prescription.

Equations (10) and (22) prescribe the kernel for the local time of arrival operator. Now we compare their results. Using the identity

$$\int_{-\infty}^{\infty} x^{-m} \exp(ix\sigma) dx = \frac{i^m \pi}{(m-1)!} \sigma^{m-1} \text{sgn}(\sigma) \quad (24)$$

to solve for the Weyl kernels of the local time of arrivals in Table-1, we find the following: For the linear and harmonic oscillator, we have the equality  $\langle q | \hat{\mathbf{T}} | q' \rangle = \langle q | \hat{\mathbf{W}}_{[t_0]} | q' \rangle$ . Thus for these cases the results of quantization and supraquantization are the same. However, for the quartic oscillator, we find the inequality  $\langle q | \hat{\mathbf{T}} | q' \rangle \neq \langle q | \hat{\mathbf{W}}_{[t_0]} | q' \rangle$ . In fact one can show that the leading term in the  $\langle q | \hat{\mathbf{T}} | q' \rangle$  of the quartic oscillator is the Weyl kernel of the same system. Because of this and equation (6) is canonically conjugate to the Hamiltonian in the sense of equation (8), the inequality  $\langle q | \hat{\mathbf{T}} | q' \rangle \neq \langle q | \hat{\mathbf{W}}_{[t_0]} | q' \rangle$  implies that the Weyl quantization of the local time of arrival for the quartic oscillator fails to be canonically conjugate with the Hamiltonian, even though the local TOA and the classical Hamiltonian are. This is another demonstration of the well known limitation of the Weyl quantization, and prescriptive quantizations in general, in satisfying required commutator values.

Our examples above demonstrate our general result for entire analytic potentials that only for linear systems that Weyl quantization is consistent and that for non-linear systems Weyl quantization only yields the leading term to the time kernel:

$$\langle q | \hat{\mathbf{T}} | q' \rangle = \begin{cases} \langle q | \hat{\mathbf{W}}_{[t_0]} | q' \rangle & : \text{ linear systems} \\ \langle q | \hat{\mathbf{W}}_{[t_0]} | q' \rangle + \Delta(q, q') & : \text{ non-linear systems} \end{cases} \quad (25)$$

where  $\Delta(q, q')$  is the correction term introduced by the non-linearity of the system.

## 6 Conclusion

In this paper we have raised the issue and the problem of constructing quantum observables without quantization. Our results demonstrate the capability of supraquantization to yield acceptable solutions where quantization fails. We recognize though that we have raised more questions in attempting to answer one. For example, is the present set of axioms of quantum mechanics sufficient to derive all of classical dynamical observables? If we strongly assert the autonomy and internal coherence of quantum mechanics, then it may be necessary to modify the present axioms to give satisfactory solution to the quantum-classical correspondence problem. An example of such modification is in the expansion of the definition of quantum observables to include positive operator valued measures [9]. Exclusion of these would lead the exclusion of numerous classical observables whose quantum image are non-self-adjoint operators. A further expansion of quantum observables would include subnormal operators. Such operators would be the appropriate quantum image of the reciprocal of classical observables whose quantum image are non-invertible, e.g. the inverse momentum for periodic boundary conditions in a segment of the real line.

Also the question of the uniqueness of the solution to the quantum-classical correspondence problem arises. As evidenced by the free particle the solution may not be unique. (See Section-3.2.) However, the non-uniqueness arises from the manner in which we take the classical limit, particularly the manner in which  $\hbar$  vanishes in the classical regime. The  $\mathcal{T}_\hbar$  transform of  $|q - q'|$ , equation (16), is to the order  $\mathcal{O}(\hbar)$  so that in the limit of vanishing  $\hbar$  it does not contribute anything. Since  $\hbar^{-1}\mu|q - q'|$  commutes with the free Hamiltonian in  $\Phi$  and it is Galilean invariant, equation (9) is arbitrary up to an additive real multiple of  $\hbar^{-1}\mu|q - q'|$ . However, if we take classical mechanics to be the limiting theory of quantum mechanics where  $\hbar$  is infinitesimal, i.e. only second and higher powers of  $\hbar$  vanish, then  $\hbar^{-1}\mu|q - q'|$  will introduce unwanted correction to the classical time of arrival. If we adhere strictly to the correspondence principle as implemented by the manner in which  $\hbar$  disappear in the classical domain,  $\hbar = \delta$ , then  $\hbar^{-1}\mu|q - q'|$  has to be dropped. This shows that supraquantization may not be independent from the manner in which classical observables are recovered from quantum observables.

Our answers above are but cursory and the questions still require greater depth than where we are at present. Moreover, there are many other questions aside from the above two (For example, how do we determine which set of axioms goes with a given problem? Is there a general framework upon which the solution to the QCC-problem without quantization can be built upon?), but right now we can only hope to address them in the future.

## Acknowledgement

The author is grateful to Prof. Kilin for the invitation to participate in the conference and appreciates the warm hospitality of E. Tolkacheva and D. Horoshko. The author acknowledges the support given by the Commission on Higher Education which made his travel possible.

## References

- [1] Zhang W and Feng D H 1995 *Phys. Rep* **252** p 1 and references therein; Brun T A et al 1997 *Phys. Let. A* **229** p 267; Trindale M and Nemes M C 1998 *Physica A* **259** p 291; Tzanakis C and Grecos A P 1998 *Physica A* **256** p 87; Ballentine L E 1999 *Phys. Let A* **261** p 145.
- [2] Kirilov, A A 1990 Geometric Quantization In: *Dynamical Systems IV: Symplectic Geometry and its Applications* Ar'nold, V I and Novikov, SP Eds. Encyclopedia Math Sci IV. (Springer, New York) p 137-172; Folland G B 1989 *Harmonic Analysis in Phase Space* Ann. Math. Ser. **122** (Princeton University Press, Princeton)
- [3] Gotay M J math-ph/9809011 and references therein.
- [4] Groenwold, H J 1946 *Physica* **12** p 405-460; van Hove, L 1951 *Proc. Roy. Acad. Sci. Belgium* **26** p 1-102.
- [5] Mackey, G W 1968 *Induced Representations of Groups and Quantum Mechanics* (W.A. Benjamin, Inc, Newyork, and Editore Boringhieri, Torino). Jauch J M 1968 *Foundations of Quantum Mechanics* (Addison-Wesley Publishing Company) p195-206.
- [6] Werner R F quant-ph/9504016 and references therein; Werner R F and Wolff M P H 1995 *Phys. Let A* **202** p 155.
- [7] Egusquiza I, Muga J G 1999 *Phys. rev. A* accepted, quant-ph 9901055 and references therein.
- [8] Leon J quant-ph/0002011.
- [9] Busch P, Grabowski M, and Lahti PJ (1995) *Operational Quantum Physics* Lecture Notes in Physics.

- [10] Weyl H 1954 *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc.) p 272-276
- [11] de Groot S R and Suttrop L G 1972 *Foundations of Electrodynamics* (North Holland Publishing Company, Amsterdam; American Elsevier Publishing Co. Inc, New York) p341-364.